

Ramsey Functions for Generalized Progressions

Mano Vikash Janardhanan & Sujith Vijay

IISER Thiruvananthapuram

manovikash@iisertvm.ac.in, sujith@iisertvm.ac.in

Abstract

Given positive integers n and k , a k -term semi-progression of scope m is a sequence (x_1, x_2, \dots, x_k) such that $x_{j+1} - x_j \in \{d, 2d, \dots, md\}$, $1 \leq j \leq k-1$, for some positive integer d . Thus an arithmetic progression is a semi-progression of scope 1. Let $S_m(k)$ denote the least integer for which every coloring of $\{1, 2, \dots, S_m(k)\}$ yields a monochromatic k -term semi-progression of scope m . We obtain an exponential lower bound on $S_m(k)$ for all $m = O(1)$. Our approach also yields a marginal improvement on the best known lower bound for the analogous Ramsey function for quasi-progressions, which are sequences whose successive differences lie in a small interval.

1. Introduction

In 1927, B.L. van der Waerden [6] proved that given positive integers r and k , there exists an integer $W(r, k)$ such that any r -coloring of $\{1, 2, \dots, W(r, k)\}$ yields a monochromatic k -term arithmetic progression. Even after nine decades, the gap between the lower and upper bounds is enormous, with the best known lower bound of the order of r^k , whereas the best known upper bound is a five-times iterated tower of exponents (see [1]). Analogues of the Van der Waerden threshold $W(r, k)$ have been studied for many variants of arithmetic progressions, including semi-progressions and quasi-progressions (see [4]).

Given positive integers m and k , a k -term semi-progression of scope m is a sequence (x_1, x_2, \dots, x_k) such that for some positive integer d , $x_{j+1} -$

$x_j \in \{d, 2d, \dots, md\}$. The integer d is called the *low-difference* of the semi-progression. We define $S_m(k)$ as the least integer for which any 2-coloring of $\{1, 2, \dots, S_m(k)\}$ yields a monochromatic k -term semi-progression of scope m . Note that $S_m(k) \leq W(k)$ with equality if $m = 1$.

2. An Exponential Lower Bound for $S_m(k)$

Landman [3] showed that $S_m(k) \geq (2k^2/m)(1 + o(1))$. We improve this to an exponential lower bound for all $m = O(1)$.

Theorem $S_m(k) > \alpha^k$ where $\alpha = \alpha(m) = \sqrt{2^m/(2^m - 1)}$

Proof Let $f(N, k, m)$ denote the number of 2-colorings of $[1, N]$ with a monochromatic k -term semiprogession of scope m . (In the remainder of the proof, we only consider k -term semi-progressions of scope m .) Note that $S_m(k)$ is the least integer N such that $f(N, k, m) = 2^N$. We derive an upper bound on $f(N, k, m)$ as follows.

Given a semi-progression $P = \{a_1, a_2, \dots, a_k\}$ of low-difference d , we define the conjugate vector of P as $(u_1, u_2, \dots, u_{k-1})$ where $u_i = (a_{i+1} - a_i - d)/d$. Likewise, the frequency vector of P is defined as $(v_0, v_1, \dots, v_{m-1})$ where v_j is the number of times j occurs in the conjugate vector of P . Finally, the weight of P , denoted $w(P)$ is defined as $u_1 + u_2 + \dots + u_{k-1}$.

Given a coloring χ , we define the (a, d) -*primary semi-progression* of χ as the semi-progression P whose conjugate vector is lexicographically least among the conjugate vectors of all semi-progressions (with first term a and low-difference d) that are monochromatic under χ . Let $P = \{a_1, a_2, \dots, a_k\}$ be a semi-progression with first term $a_1 = a$ and low-difference d . We will give an upper bound for the number of colorings χ such that P is the (a, d) -*primary semi-progression* of χ .

Since P is monochromatic, all elements of P have the same color under χ . Furthermore, if $(u_1, u_2, \dots, u_{k-1})$ is the conjugate vector of P , it follows from the fact that P is the (a, d) -primary semi-progression of χ that $w(P)$ elements in the arithmetic progression $\{a, a + d, \dots, a + m(k - 1)d\}$ must be

of the color different from the color of the elements of P . For example, let $a = 17, d = 5, m = 3, k = 6$ and $P = \{17, 32, 42, 47, 62, 72\}$ with conjugate vector $(2, 1, 0, 2, 1)$. If the two colors are red and blue, and the elements of P are all red, then 22, 27, 37, 52, 57 and 67 must all be blue. Indeed, if 57 is red, then the semi-progression $P' = \{17, 32, 42, 47, 57, 62\}$ would have a lexicographically lower conjugate vector $(2, 1, 0, 1, 0)$. Thus there are at most 2^{N-11} colorings of $[1, N]$ whose (a, d) -primary semi-progression is P .

Let $\rho = (1, 1, \dots, 1) \in \mathbb{Z}^m$ and $\mu = (0, 1, \dots, m-1)$. Clearly, $w(P) = \sum_{j=0}^{m-1} jv_j = \langle \mu, \mathbf{v} \rangle$ where \mathbf{v} is the frequency vector of P . Note that there are at most $N^2/(k-1)$ choices for the pair (a, d) . We say that two progressions P_1 and P_2 with the same a and d are equivalent if they have the same frequency vector. Note that for any a and d , there are at most

$$M(P) = \frac{(v_0 + v_1 + \dots + v_m)!}{v_0!v_1!\dots v_{m-1}!}$$

semi-progressions with the same frequency vector $(v_0, v_1, \dots, v_{m-1})$ as P . Adding over all the equivalence classes of semi-progressions, we obtain

$$f(N, k, m) \leq \frac{N^2 2^{N-k+1}}{k-1} \sum_{w(P)=0}^{(m-1)(k-1)} M(P) 2^{-w(P)}$$

It follows from the multinomial theorem that

$$f(N, k, m) \leq \frac{N^2 2^N}{k-1} \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^m} \right)^k$$

Thus $f(N, k, m) < 2^N$ for $N = \alpha_m^k$ where $\alpha_m = \sqrt{2^m/(2^m - 1)}$. This completes the proof. \blacksquare

3. Exponential Lower Bounds for $Q_n(r, k)$

We now apply the same technique to quasi-progressions. A k -term quasi-progression of low difference d and diameter n is a sequence (a_1, a_2, \dots, a_k) such that $d \leq a_{j+1} - a_j \leq d + n, 1 \leq j \leq k-1$. Let $Q_n(r, k)$ denote the least positive integer such that any r -coloring of $\{1, 2, \dots, Q_n(r, k)\}$ yields a monochromatic k -term quasi-progression of diameter n . It is known (see [5])

that $Q_1(2, k) > \beta^k$ where $\beta = 1.08226\dots$ is the smallest positive real root of the equation

$$y^{24} + 8y^{20} - 112y^{16} - 128y^{12} + 1792y^8 + 1024y^4 - 4096 = 0$$

and that $Q_n(k) = O(k^2)$ for $n > k/2$ (see [2]). We apply the techniques of the previous section to obtain lower bounds on $Q_n(r, k)$. Let $g(r, N, k, n)$ denote the number of r -colorings of $[1, N]$ with a monochromatic k -term semiprogression of diameter n . Note that $Q_n(r, k)$ is the least positive integer N such that $g(r, N, k, n) = 2^N$. We first discuss the simplest non-trivial case, namely $r = 2$ and $n = 1$.

We define the conjugate vector of a quasiprogession $Q = \{a_1, a_2, \dots, a_k\}$ of low-difference d as $(u_1, u_2, \dots, u_{k-1})$ where $u_i = a_{i+1} - a_i - d$. Given a coloring χ , we define the (a, d) -primary quasi-progression of χ as the quasi-progression Q whose conjugate vector is lexicographically least among the conjugate vectors of all quasi-progressions (with first term a and low-difference d) that are monochromatic under χ . Let $Q = \{a_1, a_2, \dots, a_k\}$ be a quasi-progression with first term $a_1 = a$ and low-difference d . We give an upper bound for the number of colorings χ such that Q is the (a, d) -primary quasi-progression of χ .

Since Q is monochromatic, all elements of Q have the same color under χ , say red. Let $(u_1, u_2, \dots, u_{k-1})$ be the conjugate vector of Q . Observe that if $u_j = 1$ and $u_{j+1} = 0$ for some j , so that $a_j, a_j + d + 1$ and $a_j + 2d + 1$ are elements of Q , and therefore red, it follows that the color of $a_j + d$ is different from red (say blue), as $(P \cup \{a_j + d\}) \setminus \{a_j + d + 1\}$ has a lexicographically lower conjugate vector. We define the weight of Q , denoted $w(Q)$, as the sum of the last element of the conjugate vector of Q , and the number of occurrences of the string “10” in the conjugate vector of Q . Note that in view of the above observation, the color of $w(Q)$ integers in the set $\{a, a+d, a+d+1, \dots, a+(k-1)d, \dots, a+(k-1)(d+1)\}$ can be inferred to be blue.

We now derive an upper bound on $g(2, N, k, 1)$. There are $N^2/(k-1)$ choices for the pair (a, d) . Of the 2^{k-1} possible conjugate vectors for a quasi-progression with first term a and common difference d , let w_ℓ be the number of conjugate

vectors of weight ℓ . Let

$$S_t = \sum_{\ell=0}^{\lceil t/2 \rceil} w_\ell 2^{-\ell}$$

denote the weighted sum of all such vectors of length t . Clearly, $S_t = S_{t,0} + S_{t,1}$ where $S_{t,0}$ and $S_{t,1}$ denote the weighted sum of conjugate vectors that begin with 0 and 1 respectively, with $S_{1,0} = 1$ and $S_{1,1} = 1/2$. It is easy to see that $A[S_{t-1,0} \ S_{t-1,1}]^T = [S_{t,0} \ S_{t,1}]^T$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1/2 & 1 \end{bmatrix}$$

Since $\lambda_{\max}(A) = 1 + \frac{1}{\sqrt{2}}$, we get

$$g(2, N, k, 1) < \frac{N^2 2^{N-k+1} \left[\left(1 + \frac{1}{\sqrt{2}}\right)^k + \left(1 - \frac{1}{\sqrt{2}}\right)^k \right]}{2(k-1)}$$

Thus $g(2, N, k, 1) < 2^N$ for $N = \beta_{2,1}^k$ where $\beta_{2,1} = 1.08239\dots$ is the smallest positive real root of the equation $y^4 - 8y^2 + 8 = 0$. It follows that $Q_1(2, k) > \beta_{2,1}^k$ yielding a marginal improvement over the lower bound in [5].

In general, since there are r^N r -colorings of $[1, N]$ and at most $N^2(n+1)^{k-1}$ k -term quasi-progressions of diameter n , a lower bound of the form $Q_n(r, k) \geq (\sqrt{r/(n+1)})^k$ follows immediately from the linearity of expectation. However, this bound is only useful when $n \leq r - 2$. Generalising the approach outlined earlier, we represent the conjugate vector of Q as an r -ary string, and define the weight $w(Q)$ as the sum of the last element of the conjugate vector of Q , and the number of occurrences of strings of length two of the form “ xy ”, counted with multiplicity $m(x, y) = \min(x, n - y)$. (Note that $m(x, y)$ denotes the number of conjugate vectors that are lexicographically lower than the given vector and correspond to quasi-progressions that differ from Q in exactly one element.) As before, let $S_{t,j}$ denote the weighted sum of conjugate vectors of length t beginning with j , $0 \leq j \leq n$, with $S_{1,j} = \alpha^j$ for all j where $\alpha = 1 - \frac{1}{r}$. Then

$A[S_{t,0} \cdots S_{t,n}]^T = [S_{t+1,0} \cdots S_{t+1,n}]^T$ where

$$A_{r,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha & \alpha & \cdots & \alpha & 1 \\ \alpha^2 & \alpha^2 & \cdots & \alpha & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^n & \alpha^{n-1} & \cdots & \alpha & 1 \end{bmatrix}$$

Then $Q_n(r, k) > \beta^k$ where $\beta = \beta_{r,n} = \sqrt{r/\lambda_{\max}(A_{r,n})}$. Note that for each r , there are only finitely many values for which $\beta_{r,n} > 1$. The first few such values are shown in the following table.

n	1	2	3	4	5	6
$\beta_{2,n}$	1.08239	< 1	< 1	< 1	< 1	< 1
$\beta_{3,n}$	1.28511	1.11226	1.02236	< 1	< 1	< 1
$\beta_{4,n}$	1.46410	1.24686	1.12770	1.05338	1.00384	< 1

References

- [1] W. T. Gowers, *A new proof of Szemerédi's theorem*. Geometric and Functional Analysis 11, 2001.
- [2] A. Jobson, A. Kezdy, H. Snevily and S. C. White, *Ramsey functions for quasi-progressions with large diameter*. Journal of Combinatorics 2 (2011), 557-573.
- [3] B. M. Landman, *Monochromatic sequences whose gaps belong to $\{d, 2d, \dots, md\}$* . Bulletin of the Australian Mathematical Society 58 (1998), 93-101.
- [4] B. M. Landman and A. Robertson, *Ramsey Theory on the Integers*. American Mathematical Society, Providence, 2004.
- [5] S. Vijay, *On a variant of Van der Waerden's Theorem*. Integers 10 (2010), A17, 5pp. (electronic).
- [6] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*. Nieuw Archief voor Wiskunde 15 (1927), 212-216.